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Related Topics:Differential EquationsMathematics \(\rightarrow\) Differential Equations \(\rightarrow\) Nonlinear Differential Equations Description: Nonlinear differential equations form a subfield of differential equations, which is a core area within mathematics. While differential equations in general are equations that involve functions and their derivatives, nonlinear differential equations are characterized by the presence of nonlinearity in these relationships. This nonlinearity can occur in several forms, such as products of the function and its derivatives, powers of these terms, or other non-linear combinations. Characteristics and Definition A differential equation is categorized as nonlinear if it cannot be written as a linear combination of the dependent variable and its derivatives. In general form, a nonlinear differential equation can be represented as: \[F \left(x, y, y', y'', \ldots, y^{(n)} \right) = 0, \] where \(F \) is a nonlinear function of the independent variable \(x \), the dependent variable \(y \), and its derivatives \(y', y'', \ldots, y^{(n)} \). In contrast to linear differential equations, whose solutions can often be superimposed to form new solutions, nonlinear differential equations exhibit much more complex behavior. Importance and Applications The study of nonlinear differential equations is crucial due to their widespread applications in various scientific and engineering fields such as physics, biology, chemistry, economics, and more. Many natural phenomena, including fluid dynamics, electrical circuits, population dynamics, and chaos theory, are governed by nonlinear differential equations. Because of their complexity, these equations often require special methods for their study and solution. Solution Techniques Solving nonlinear differential equations can be significantly more challenging compared to linear ones. Exact solutions are rare and often not possible, thus numerous analytical and numerical techniques are employed: Perturbation Methods: These involve approximating the solution by introducing a small parameter, allowing the nonlinear problem to be treated as a series of simpler problems. Phase Plane Analysis: This geometric approach involves studying the trajectories of systems of first-order nonlinear differential equations in a plane, providing qualitative insights into the behavior of solutions. Lyapunov Methods: Used primarily in stability analysis, Lyapunov functions help assess whether the solutions of a nonlinear system remain close to an equilibrium point over time. Numerical Methods: Techniques such as the Runge-Kutta method, finite difference method, and shooting method are frequently used for obtaining approximate solutions to nonlinear differential equations. Examples 1. Logistic Growth Equation: \[\frac{dy}{dt} = ry \left(1 - \frac{y}{K} \right), \] where \(r \) is the growth rate and \(K \) is the carrying capacity. This equation models population growth limited by resources. 2. The Van der Pol Oscillator: \[\frac{d^2 x}{dt^2} - \mu (1 - x^2) \frac{dx}{dt} + x = 0, \] where \(\mu \) is a parameter indicating the nonlinearity and the strength of the damping. Challenges and Future Directions The inherent complexity and diverse behaviors of nonlinear differential equations pose ongoing challenges to mathematicians and scientists. Developing new analytical and numerical methods is an active area of research. Understanding and predicting the behavior of nonlinear systems better can lead to significant advancements in science and technology. In summary, nonlinear differential equations represent a fascinating and vital area of mathematics, owing to their complex nature and the myriad of applications across different fields. As we continue to make strides in this domain, we gain deeper insights into the intricacies of natural and engineered systems. A differential equation is an equation that relates one or more unknown functions and their derivatives. This type of equation can take on many different forms. The functions in an application typically stand in for physical quantities, the derivatives for the rates of change in those values, and the differential equation for defining the relationship between the two. Let's start with the basics and learn what a linear differential equation is before moving on to the more complicated topic of a non-linear differential equation.Linear Equation Differential: A differential equation that is defined by a linear polynomial in the unknown function and its derivatives is called a linear differential equation.The general form of a linear equation is a0 (x) y + a1 (x) y' + a2 (x) y'' + … + an (x) yn = b (x)where a0(x), …, an(x) and b(x) are arbitrary differentiable functions that don't have to be linear, and y',…, y(n) are the successive derivatives of an unknown function y of the variable x.This kind of equation is called an ordinary differential equation. If the unknown function depends on more than one variable and the derivatives in the equation are partial derivatives, then the linear differential equation can also be called a linear partial differential equation.Nonlinear Differential EquationA non-linear differential equation is one in which the unknown function and its derivatives don't have a straight line when plotted in a graph (the linearity or non-linearity in the arguments of the function are not considered here). There aren't many ways to solve nonlinear differential equations exactly, and the ones that do exist usually require the equation to have certain symmetries.Over long periods of time, nonlinear differential equations can act in very strange ways. This is a sign of chaos. Even the most basic questions about the existence, uniqueness and extendability of solutions for nonlinear differential equations and the well-posedness of initial and boundary value problems for nonlinear partial differential equations are hard to answer, and when they are, it is considered a big step forward in mathematical theory. But if the differential equation is a correct description of a real-world physical process, then it should have a solution.Differential equations that are linear are often used to approximate equations that are not linear. These are only close estimates that work in certain situations.For instance, the harmonic oscillator equation is a close approximation of the nonlinear pendulum equation that works for oscillations with small amplitudes.There are some nonlinear differential equations for which the exact solutions are known, but many of those that are essential in applications do not have known exact solutions. Occasionally, these equations can be linearized using a procedure known as expansion, in which the nonlinear elements are eliminated. This cannot be done in situations where nonlinear terms provide significant contributions to the solution; nevertheless, there are occasions when it is sufficient to keep a few "small" ones. General FormA nonlinear differential equation is an equation of the formxn+1 = f(xn, xn-1,…)where x„ is the value of x in generation n and where the recursion function f depends on nonlinear combinations of its arguments (f may involve quadratics, exponentials, reciprocals, or powers of the x„'s, and so forth). A solution is again a general formula relating x„ to the generation n and to some initially specified values, e.g., x0, x1, and so on.Difference Between Linear and Nonlinear EquationsIn the first set of examples, u is a function of x that we don't know, while c is known constants. Ordinary differential equations and partial differential equations can be put into two large groups: linear differential equations and nonlinear differential equations. Also, differential equations can be divided into homogeneous and heterogeneous differential equations.The ordinary differential equation for a first-order linear system with a constant coefficient is du/dx = cu + x^2Homogeneous second-order linear ordinary differential equation is (d^2u/dx^2) – x(du/dx) + u = $0(d^2u/dx^2) + \omega^2u = 0$ is the ordinary differential equation for the harmonic oscillator with a homogeneous second-order linear constant coefficient.du/dx = u^2 + 4 is the first-order nonlinear heterogeneous ordinary differential equation.The motion of a pendulum of length L is described by the equation L (d^2u/dx^2) + g sin u = 0This equation is nonlinear because of the sine function.In the next set of examples, the unknown function u depends on two variables, x and t or x and y.(∂u/∂t) + t(∂u/∂x) = 0 is a homogeneous first-order linear partial differential equation.$(\partial^2u/\partial x^2) + (\partial^2u/\partial y^2) = 0$ is the Laplace equation for a homogeneous second-order linear constant coefficient elliptic type partial differential equation.Homogeneous third-order non-linear partial differential equation(∂u/∂t) = 6u(∂u/∂x) − $(\partial^3u/\partial x^3)$ConclusionA nonlinear differential equation is one that is not linear with respect to the unknown function and its derivatives. Linear differential equations are frequently used to approximate nonlinear equations. These are simply approximations that work in some circumstances. The general form of a nonlinear differential equation is xn+1 = f(xn, xn-1,…). A harmonic oscillator equation is an approximation of the nonlinear pendulum equation that is valid for oscillations of modest amplitude and is an example of non-linear differential equations. A differential equation is an equation that relates one or more unknown functions and their derivatives. This type of equation can take on many different forms. The functions in an application typically stand in for physical quantities, the derivatives for the rates of change in those values, and the differential equation for defining the relationship between the two. 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Type of ordinary differential equation Differential equations Scope Fields Natural sciencesEngineering Astronomy Physics Chemistry Biology Geology Applied mathematics Continuum mechanics Chaos theory Dynamical systems Social sciences Economics Population dynamics List of named differential equations Classification Types Ordinary Partial Differential-algebraic Integro-differential Fractional Linear Non-linear By variable type Dependent and independent variables Autonomous Coupled / Decoupled Exact Homogeneous / Nonhomogeneous Features Order Operator Notation Relation to processes Difference (discrete analogue) Stochastic Stochastic partial Delay Solution Existence and uniqueness Picard–Lindelöf theorem Peano existence theorem Carathéodory's existence theorem Cauchy–Kowalevski theorem General topics Initial conditions Boundary values Dirichlet Neumann Robin Cauchy problem Wronskian Phase portrait Lyapunov / Asymptotic / Exponential stability Rate of convergence Series / Integral solutions Numerical integration Dirac delta function Solution methods Inspection Method of characteristics Euler Exponential response formula Finite difference (Crank–Nicolson) Finite element Infinite element Finite volume Galerkin Petrov–Galerkin Green's function Integrating factor Integral transforms Perturbation theory Runge–Kutta Separation of variables Undetermined coefficients Variation of parameters People List Isaac Newton Gottfried Leibniz Jacob Bernoulli Leonhard Euler Joseph-Louis Lagrange Józef Maria Hoene-Wroński Joseph Fourier Augustin-Louis Cauchy George Green Carl David Tolmé Runge Martin Kutta Rudolf Lipschitz Ernst Lindelöf Émile Picard Phyllis Nicolson John Crank vte In mathematics, an ordinary differential equation is called a Bernoulli differential equation if it is of the form y ′ + P (x) y = Q (x) y n , {\displaystyle y'+P(x)y=Q(x)y^{n},} where n {\displaystyle n} is a real number. Some authors allow any real n {\displaystyle n} ,[1][2] whereas others require that n {\displaystyle n} not be 0 or 1.[3][4] The equation was first discussed in a work of 1695 by Jacob Bernoulli, after whom it is named. The earliest solution, however, was offered by Gottfried Leibniz, who published his result in the same year and whose method is the one still used today.[5] Bernoulli equations are special because they are nonlinear differential equations with known exact solutions. A notable special case of the Bernoulli equation is the logistic differential equation. When n = 0 {\displaystyle n=0} , the differential equation is linear. When n = 1 {\displaystyle n=1} , it is separable. In these cases, standard techniques for solving equations of those forms can be applied. For n ≠ 0 {\displaystyle neq 0} and n ≠ 1 {\displaystyle neq 1} , the substitution u = y 1 − n {\displaystyle u=y^{1-n}} reduces any Bernoulli equation to a linear differential equation d u d x − (n − 1) P (x) u = − (n − 1) Q (x) . {\displaystyle {\frac {du}{dx}}-(n-1)P(x)u=-(n-1)Q(x).} For example, in the case n = 2 {\displaystyle n=2} , making the substitution u = y − 1 {\displaystyle u=y^{-1}} in the differential equation d y d x + 1 x y = x y 2 {\displaystyle {\frac {dy}{dx}}+{\frac {1}{x}}y=xy^{2}} produces the equation d u d x − 1 x u = − x {\displaystyle {\frac {du}{dx}}-{\frac {1}{x}}u=-x} , which is a linear differential equation. Let x 0 ∈ (a , b) {\displaystyle x_{0}\in (a,b)} and { z : (a , b) → (0 , ∞) , if α ∈ R ∖ { 1 , 2 } , z : (a , b) → R ∖ { 0 } , if α = 2 , {\displaystyle {\begin{cases}z:(a,b)\rightarrow (0,\infty),&{\text{if }}\alpha \in \mathbb {R} \smallsetminus \{1,2\},\\[4pt]z:(a,b)\rightarrow \mathbb {R} \smallsetminus \{0\},&{\text{if }}\alpha =2,\end{cases}}} be a solution of the linear differential equation z ′ (x) = (1 − α) P (x) z (x) + (1 − α) Q (x) . {\displaystyle z'(x)=(1-\alpha)P(x)z(x)+(1-\alpha)Q(x).} Then we have that y (x) := [z (x)] 1 / (1 − α) {\displaystyle y(x):=[z(x)]^{1/(1-\alpha)}} is a solution of y ′ (x) = P (x) y (x) + Q (x) y α (x)   ,   y (x 0) = y 0 := [z (x 0)] 1 / (1 − α) . {\displaystyle y'(x)=P(x)y(x)+Q(x)y^{\alpha }(x)\ ,\ y(x_{0})=y_{0}:=[z(x_{0})]^{1/(1-\alpha)}.} And for every such differential equation, for all α > 0 {\displaystyle \alpha >0} we have y ≡ 0 {\displaystyle y\equiv 0} as solution for y 0 = 0 {\displaystyle y_{0}=0} . Consider the Bernoulli equation y ′ − 2 y x = − x 2 y 2 {\displaystyle y'-{\frac {2y}{x}}=-x^{2}y^{2}} (in this case, more specifically a Riccati equation). The constant function y = 0 {\displaystyle y=0} is a solution. Division by y 2 {\displaystyle y^{2}} yields y ′ y − 2 − 2 x y − 1 = − x 2 {\displaystyle y'y^{-2}-{\frac {2}{x}}y^{-1}=-x^{2}} Changing variables gives the equations u = 1 y ,   u ′ = − y ′ y 2 − u ′ − 2 x u = − x 2 u ′ + 2 x u = x 2 {\displaystyle {\begin{aligned}u={\frac {1}{y}}\;&,~u'={\frac {-y'}{y^{2}}}\\[5pt]-u'-{\frac {2}{x}}u&=-x^{2}\\[5pt]u'+{\frac {2}{x}}u&=x^{2}\end{aligned}}} which can be solved using the integrating factor M (x) = e 2 ∫ 1 x d x = e 2 ln x = x 2 . {\displaystyle M(x)=e^{2\int {\frac {1}{x}}\,dx}=e^{2\ln x}=x^{2}.} Multiplying by M (x) {\displaystyle M(x)} , u ′ x 2 + 2 x u = x 4 . {\displaystyle u'x^{2}+2xu=x^{4}.} The left side can be represented as the derivative of u x 2 {\displaystyle ux^{2}} by reversing the product rule. Applying the chain rule and integrating both sides with respect to x {\displaystyle x} results in the equations ∫ (u x 2) ′ d x = ∫ x 4 d x u x 2 = 1 5 x 5 + C 1 y x 2 = 1 5 x 5 + C {\displaystyle {\begin{aligned}\int \left(ux^{2}\right)'dx&=\int x^{4}\,dx\\[5pt]ux^{2}&={\frac {1}{5}}x^{5}+C\\[5pt]{\frac {1}{y}}x^{2}&={\frac {1}{5}}x^{5}+C\end{aligned}}} The solution for y {\displaystyle y} is y = x 2 1 5 x 5 + C . {\displaystyle y={\frac {x^{2}}{{\frac {1}{5}}x^{5}+C}}.} ^ Zill, Dennis G. (2013). A First Course in Differential Equations with Modeling Applications (10th ed.). Boston, Massachusetts: Cengage Learning. p. 73. ISBN 9780357088364. ^ Stewart, James (2015). Calculus: Early Transcendentals (8th ed.). Boston, Massachusetts: Cengage Learning. p. 625. ISBN 9781305482463. ^ Rozov, N. Kh. (2001) [1994], "Bernoulli equation", Encyclopedia of Mathematics, EMS Press ^ Teschl, Gerald (2012). "1.4. Finding explicit solutions" (PDF). Ordinary Differential Equations and Dynamical Systems. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society. p. 15. eISSN 2376-9203. ISBN 978-0-8218-8328-0. ISSN 1065-7339. Zbl 1263.34002. ^ Parker, Adam E. (2013). "Who Solved the Bernoulli Differential Equation and How Did They Do It?" (PDF). The College Mathematics Journal. 44 (2): 89–97. ISSN 2159-8118 – via Mathematical Association of America. Bernoulli, Jacob (1695), "Explicationes, Annotationes & Additiones ad ea, quae in Actis sup. de Curva Elastica, Isochrona Paracentrica, & Velaria, hinc inde memorata, & paratim controversa legundur; ubi de Linea mediarum directionum, alliisque novis", Acta Eruditorum. Cited in Hairer, Nørsett & Wanner (1993). Hairer, Ernst; Nørsett, Syvert Paul; Wanner, Gerhard (1993), Solving ordinary differential equations I: Nonstiff problems, Berlin, New York: Springer-Verlag, ISBN 978-3-540-56670-0. Index of differential equations Retrieved from " In mathematics, the binomial differential equation is an ordinary differential equation of the form (y ′) m = f (x , y) , {\displaystyle \left(y'\right)^{m}=f(x,y),} where m {\displaystyle m} is a natural number and f (x , y) {\displaystyle f(x,y)} is a polynomial that is analytic in both variables.[1][2] Let P (x , y) = (x + y) k {\displaystyle P(x,y)=(x+y)^{k}} be a polynomial of two variables of order k {\displaystyle k} , where k {\displaystyle k} is a natural number. By the binomial formula, P (x , y) = ∑ j = 0 k (k j) x j y k − j {\displaystyle P(x,y)=\sum \limits _{j=0}^{k}{{\binom {k}{j}}x^{j}y^{k-j}}} . [relevant?] The binomial differential equation becomes (y ′) m = (x + y) k {\textstyle (y')^{m}=(x+y)^{k}} .[clarification needed] Substituting v = x + y {\displaystyle v=x+y} and its derivative v ′ = 1 + y ′ {\displaystyle v'=1+y'} gives (v ′ − 1) m = v k {\textstyle (v'-1)^{m}=v^{k}} , which can be written d v d x = 1 + v k m {\textstyle {\tfrac {dv}{dx}}=1+v^{\tfrac {k}{m}}} , which is a separable ordinary differential equation. Solving gives d v d x = 1 + v k m ⇒ d v 1 + v k m = d x ⇒ ∫ d v 1 + v k m = x + C {\displaystyle {\begin{array}{lrl}&{\frac {dv}{dx}}&=1+v^{\tfrac {k}{m}}\\\Rightarrow &{\frac {dv}{1+v^{\tfrac {k}{m}}}}&=dx\\\Rightarrow &\int {\frac {dv}{1+v^{\tfrac {k}{m}}}}&=x+C\end{array}}} If m = k {\displaystyle m=k} , this gives the differential equation v ′ − 1 = v {\displaystyle v'-1=v} and the solution is y (x) = C e x − x − 1 {\displaystyle y\left(x\right)=Ce^{x}-x-1} , where C {\displaystyle C} is a constant. If m | k {\displaystyle m|k} (that is, m {\displaystyle m} is a divisor of k {\displaystyle k}), then the solution has the form ∫ d v 1 + v n = x + C {\textstyle \int {\frac {dv}{1+v^{n}}}=x+C} . In the tables book Gradshteyn and Ryzhik, this form decomposes as: ∫ d v 1 + v n = { − 2 n ∑ i = 0 n 2 − 1 P i cos (2 i + 1 n π) + 2 n ∑ i = 0 n 2 − 1 Q i sin (2 i + 1 n π) , n : even integer 1 n ln (1 + v) − 2 n ∑ i = 0 n − 3 2 P i cos (2 i + 1 n π) + 2 n ∑ i = 0 n − 3 2 Q i sin (2 i + 1 n π) , n : odd integer {\displaystyle \int {\frac {dv}{1+v^{n}}}=\left\{{\begin{array}{ll}-{\frac {2}{n}}\sum \limits _{i=0}^{{\textstyle {n \over 2}}-1}{P_{i}\cos \left({{\frac {2i+1}{n}}\pi }\right)}+{\frac {2}{n}}\sum \limits _{i=0}^{{\tfrac {n}{2}}-1}{Q_{i}\sin \left({{\frac {2i+1}{n}}\pi }\right)},&n:{\text{even integer}}\\\\{\frac {1}{n}}\ln \left({1+v}\right)-{\frac {2}{n}}\sum \limits _{i=0}^{\textstyle {{n-3} \over 2}}{P_{i}\cos \left({{\frac {2i+1}{n}}\pi }\right)}+{\frac {2}{n}}\sum \limits _{i=0}^{\tfrac {n-3}{2}}{Q_{i}\sin \left({{\frac {2i+1}{n}}\pi }\right)},&n:{\text{odd integer}}\\\end{array}}\right.} where P i = 1 2 ln (v 2 − 2 v cos (2 i + 1 n π) + 1) Q i = arctan (v − cos (2 i + 1 n π) sin (2 i + 1 n π)) {\displaystyle {\begin{aligned}P_{i}&={\frac {1}{2}}\ln \left({v^{2}-2v\cos \left({{\frac {2i+1}{n}}\pi }\right)+1}\right)\\Q_{i}&=\arctan \left({\frac {v-\cos \left({{\textstyle {2i+1} \over n}}\pi }\right)}{\sin \left({{\textstyle {2i+1} \over n}}\pi }\right)}}\right)\end{aligned}}} Examples of differential equations ^ Hille, Einar (1894). Lectures on ordinary differential equations. Addison-Wesley Publishing Company. p. 675. ISBN 978-0201530834. {{cite book}}: ISBN / Date incompatibility (help) ^ Zwillinger, Daniel (1998). Handbook of differential equations (3rd ed.). San Diego, Calif.: Academic Press. p. 180. ISBN 978-0-12-784396-4. Retrieved from "