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Conjecture in math

Proposition in mathematics that is unproven For text reconstruction, see Conjecture (textual criticism). The real part (red) and imaginary part (blue) of the Riemann zeta function along the critical line Re(s) = 1/2. The first non-trivial zeros can be seen at Im(s) = ±14.135, ±21.022 and ±25.011. The Riemann hypothesis, a famous conjecture, says that all non-trivial zeros of the zeta function lie along the critical line. In mathematics, a conjecture is a conclusion or a proposition that is proffered on a tentative basis without proof.[1][2][3] Some conjectures, such as the Riemann hypothesis or Fermat's conjecture (now a theorem, proven in 1995 by Andrew Wiles), have shaped much of mathematical history as new areas of mathematics are developed in order to prove them.[4] Formal mathematics is based on provable truth. In mathematics, any number of cases supporting a universally quantified conjecture, no matter how large, is insufficient for establishing the conjecture's veracity, since a single counterexample could immediately bring down the conjecture. Mathematical journals sometimes publish the minor results of research teams having extended the search for a counterexample farther than previously done. For instance, the Collatz conjecture, which concerns whether or not certain sequences of integers terminate, has been tested for all integers up to 1.2 × 1012 (1.2 trillion). However, the failure to find a counterexample after extensive search does not constitute a proof that the conjecture is true—because the conjecture might be false but with a very large minimal counterexample. Nevertheless, mathematicians often regard a conjecture as strongly supported by evidence even though not yet proved. That evidence may be of various kinds, such as verification of consequences of it or strong interconnections with known results.[5] A conjecture is considered proven only when it has been shown that it is logically impossible for it to be false. There are various methods of doing so; see methods of mathematical proof for more details. One method of proof, applicable when there are only a finite number of cases that could lead to counterexamples, is known as "brute force": in this approach, all possible cases are considered and shown not to give counterexamples. In some occasions, the number of cases is quite large, in which case a brute-force proof may require as a practical matter the use of a computer algorithm to check all the cases. For example, the validity of the 1976 and 1997 brute-force proofs of the four color theorem by computer was initially doubted, but was eventually confirmed in 2005 by theorem-proving software. When a conjecture has been proven, it is no longer a conjecture but a theorem. Many important theorems were once conjectures, such as the Geometrization theorem (which resolved the Poincaré conjecture), Fermat's Last Theorem, and others. Conjectures disproven through counterexample are sometimes referred to as false conjectures (cf. the Pólya conjecture and Euler's sum of powers conjecture). In the case of the latter, the first counterexample found for the n=4 case involved numbers in the millions, although it has been subsequently found that the minimal counterexample is actually smaller. Not every conjecture ends up being proven true or false. The continuum hypothesis, which tries to ascertain the relative cardinality of certain infinite sets, was eventually shown to be independent from the generally accepted set of Zermelo–Fraenkel axioms of set theory. It is therefore possible to adopt this statement, or its negation, as a new axiom in a consistent manner (much as Euclid's parallel postulate can be taken either as true or false in an axiomatic system for geometry). In this case, if a proof uses this statement, researchers will often look for a new proof that does not require the hypothesis (in the same way that it is desirable that statements in Euclidean geometry be proved using only the axioms of neutral geometry, i.e. without the parallel postulate). The one major exception to this in practice is the axiom of choice, as the majority of researchers usually do not worry whether a result requires it—unless they are studying this axiom in particular. Sometimes, a conjecture is called a hypothesis when it is used frequently and repeatedly as an assumption in proofs of other results. For example, the Riemann hypothesis is a conjecture from number theory that — amongst other things — makes predictions about the distribution of prime numbers. Few number theorists doubt that the Riemann hypothesis is true. In fact, in anticipation of its eventual proof, some have even proceeded to develop further proofs which are contingent on the truth of this conjecture. These are called conditional proofs: the conjectures assumed appear in the hypotheses of the theorem, for the time being. These "proofs", however, would fall apart if it turned out that the hypothesis was false, so there is considerable interest in verifying the truth or falsity of conjectures of this type. Main article: Fermat's Last Theorem In number theory, Fermat's Last Theorem (sometimes called Fermat's conjecture, especially in older texts) states that no three positive integers a {\displaystyle a} , b {\displaystyle b} , and c {\displaystyle c} can satisfy the equation a n + b n = c n {\displaystyle a^{n}+b^{n}=c^{n}} for any integer value of n {\displaystyle n} greater than two. This theorem was first conjectured by Pierre de Fermat in 1637 in the margin of a copy of Arithmetica, where he claimed that he had a proof that was too large to fit in the margin.[6] The first successful proof was released in 1994 by Andrew Wiles, and formally published in 1995, after 358 years of effort by mathematicians. The unsolved problem stimulated the development of algebraic number theory in the 19th century, and the proof of the modularity theorem in the 20th century. It is among the most notable theorems in the history of mathematics, and prior to its proof it was in the Guinness Book of World Records for "most difficult mathematical problems".[7] Main article: Four color theorem A four-coloring of a map of the states of the United States (ignoring lakes). In mathematics, the four color theorem, or the four color map theorem, states that given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map—so that no two adjacent regions have the same color. Two regions are called adjacent if they share a common boundary that is not a corner, where corners are the points shared by three or more regions.[8] For example, in the map of the United States of America, Utah and Arizona are adjacent, but Utah and New Mexico, which only share a point that also belongs to Arizona and Colorado, are not. Möbius mentioned the problem in his lectures as early as 1840.[9] The conjecture was first proposed on October 23, 1852[10] when Francis Guthrie, while trying to color the map of counties of England, noticed that only four different colors were needed. The five color theorem, which has a short elementary proof, states that five colors suffice to color a map and was proven in the late 19th century;[11] however, proving that four colors suffice turned out to be significantly harder. A number of false proofs and false counterexamples have appeared since the first statement of the four color theorem in 1852. The four color theorem was ultimately proven in 1976 by Kenneth Appel and Wolfgang Haken. It was the first major theorem to be proved using a computer. Appel and Haken's approach started by showing that there is a particular set of 1,936 maps, each of which cannot be part of a smallest-sized counterexample to the four color theorem (i.e., if they did appear, one could make a smaller counter-example). Appel and Haken used a special-purpose computer program to confirm that each of these maps had this property. Additionally, any map that could potentially be a counterexample must have a portion that looks like one of these 1,936 maps. Showing this with hundreds of pages of hand analysis, Appel and Haken concluded that no smallest counterexample exists because any must contain, yet do not contain, one of these 1,936 maps. This contradiction means there are no counterexamples at all and that the theorem is therefore true. Initially, their proof was not accepted by mathematicians at all because the computer-assisted proof was infeasible for a human to check by hand.[12] However, the proof has since then gained wider acceptance, although doubts still remain.[13] Main article: Hauptvermutung The Hauptvermutung (German for main conjecture) of geometric topology is the conjecture that any two triangulations of a triangulable space have a common refinement, a single triangulation that is a subdivision of both of them. It was originally formulated in 1908, by Steinitz and Tietze.[14] This conjecture is now known to be false. The non-manifold version was disproved by John Milnor[15] in 1961 using Reidemeister torsion. The manifold version is true in dimensions m ≤ 3. The cases m = 2 and 3 were proved by Tibor Radó and Edwin E. Moise[16] in the 1920s and 1950s, respectively. Main article: Weil conjectures In mathematics, the Weil conjectures were some highly influential proposals by André Weil (1949) on the generating functions (known as local zeta-functions) derived from counting the number of points on algebraic varieties over finite fields. A variety V over a finite field with q elements has a finite number of rational points, as well as points over every finite field with qk elements containing that field. The generating function has coefficients derived from the numbers Nk of points over the (essentially unique) field with qk elements. Weil conjectured that such zeta-functions should be rational functions, should satisfy a form of functional equation, and should have their zeroes in restricted places. The last two parts were quite consciously modeled on the Riemann zeta function and Riemann hypothesis. The rationality was proved by Dwork (1960), the functional equation by Grothendieck (1965), and the analogue of the Riemann hypothesis was proved by Deligne (1974). Main article: Poincaré conjecture In mathematics, the Poincaré conjecture is a theorem about the characterization of the 3-sphere, which is the hypersphere that bounds the unit ball in four-dimensional space. The conjecture states that: Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. An equivalent form of the conjecture involves a coarser form of equivalence than homeomorphism called homotopy equivalence: if a 3-manifold is homotopy equivalent to the 3-sphere, then it is necessarily homeomorphic to it. Originally conjectured by Henri Poincaré in 1904, the theorem concerns a space that locally looks like ordinary three-dimensional space but is connected, finite in size, and lacks any boundary (a closed 3-manifold). The Poincaré conjecture claims that if such a space has the additional property that each loop in the space can be continuously tightened to a point, then it is necessarily a three-dimensional sphere. An analogous result has been known in higher dimensions for some time. After nearly a century of effort by mathematicians, Grigori Perelman presented a proof of the conjecture in three papers made available in 2002 and 2003 on arXiv. The proof followed on from the program of Richard S. Hamilton to use the Ricci flow to attempt to solve the problem. Hamilton later introduced a modification of the standard Ricci flow, called Ricci flow with surgery to systematically excise singular regions as they develop, in a controlled way, but was unable to prove this method "converged" in three dimensions.[17] Perelman completed this portion of the proof. Several teams of mathematicians have verified that Perelman's proof is correct. The Poincaré conjecture, before being proven, was one of the most important open questions in topology. Main article: Riemann hypothesis In mathematics, the Riemann hypothesis, proposed by Bernhard Riemann (1859), is a conjecture that the non-trivial zeros of the Riemann zeta function all have real part 1/2. The name is also used for some closely related analogues, such as the Riemann hypothesis for curves over finite fields. The Riemann hypothesis implies results about the distribution of prime numbers. Along with suitable generalizations, some mathematicians consider it the most important unresolved problem in pure mathematics.[18] The Riemann hypothesis, along with the Goldbach conjecture, is part of Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems; it is also one of the Clay Mathematics Institute Millennium Prize Problems. Main article: P versus NP problem The P versus NP problem is a major unsolved problem in computer science. Informally, it asks whether every problem whose solution can be quickly verified by a computer can also be quickly solved by a computer; it is widely conjectured that the answer is no. It was essentially first mentioned in a 1956 letter written by Kurt Gödel to John von Neumann. Gödel asked whether a certain NP-complete problem could be solved in quadratic or linear time.[19] The precise statement of the P=NP problem was introduced in 1971 by Stephen Cook in his seminal paper "The complexity of theorem proving procedures"[20] and is considered by many to be the most important open problem in the field.[21] It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US$1,000,000 prize for the first correct solution. Goldbach's conjecture The twin prime conjecture The Collatz conjecture The Manin conjecture The Maldacena conjecture The Euler conjecture, proposed by Euler in the 18th century but for which counterexamples for a number of exponents (starting with n=4) were found beginning in the mid 20th century The Hardy-Littlewood conjectures are a pair of conjectures concerning the distribution of prime numbers, the first of which expands upon the aforementioned twin prime conjecture. Neither one has either been proven or disproven, but it has been proven that both cannot simultaneously be true (i.e., at least one must be false). It has not been proven which one is false, but it is widely believed that the first conjecture is true and the second one is false.[22] The Langlands program[23] is a far-reaching web of these ideas of 'unifying conjectures' that link different subfields of mathematics (e.g. between number theory and representation theory of Lie groups). Some of these conjectures have since been proved. Karl Popper pioneered the use of the term "conjecture" in scientific philosophy.[24] Conjecture is related to hypothesis, which in science refers to a testable conjecture. Bold hypothesis Futures studies Hypotheticals List of conjectures Ramanujan machine ^ "Definition of CONJECTURE". www.merriam-webster.com. Retrieved 2019-11-12. ^ Oxford Dictionary of English (2010 ed.). ^ Schwartz, JL (1995). Shuttling between the particular and the general: reflections on the role of conjecture and hypothesis in the generation of knowledge in science and mathematics. Oxford University Press. p. 93. ISBN 9780195115772. ^ Weisstein, Eric W. "Fermat's Last Theorem". mathworld.wolfram.com. Retrieved 2019-11-12. ^ Franklin, James (2016). "Logical probability and the strength of mathematical conjectures" (PDF). Mathematical Intelligencer. 38 (3): 14–19. doi:10.1007/s00283-015-9612-3. S2CID 30291085. Archived (PDF) from the original on 2017-03-09. Retrieved 30 June 2021. ^ Ore, Oystein (1988) [1948], Number Theory and Its History, Dover, pp. 203–204, ISBN 978-0-486-65620-5 ^ "Science and Technology". The Guinness Book of World Records. Guinness Publishing Ltd. 1995. ^ Georges Gonthier (December 2008). "Formal Proof—The Four-Color Theorem". Notices of the AMS. 55 (11): 1382–1393. From this paper: Definitions: A planar map is a set of pairwise disjoint subsets of the plane, called regions. A simple map is one whose regions are connected open sets. Two regions of a map are adjacent if their respective closures have a common point that is not a corner of the map. A point is a corner of a map if and only if it belongs to the closures of at least three regions. Theorem: The regions of any simple planar map can be colored with only four colors, in such a way that any two adjacent regions have different colors. ^ W. W. 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(June 2016) (Learn how and when to remove this message) Stereographic projection of the hypersphere's parallels (red), meridians (blue) and hypermeridians (green). Because this projection is conformal, the curves intersect each other orthogonally (in the yellow points) as in 4D. All curves are circles: the curves that intersect ⟨0,0,0,1⟩ have infinite radius (= straight line). In this picture, the whole 3D space maps the surface of the hypersphere, whereas in the next picture the 3D space contained the shadow of the bulk hypersphere. Direct projection of 3-sphere into 3D space and covered with surface grid, showing structure as stack of 3D spheres (2-spheres) In mathematics, a hypersphere or 3-sphere is a 4-dimensional analogue of a sphere, and is the 3-dimensional n-sphere. In 4-dimensional Euclidean space, it is the set of points equidistant from a fixed central point. The interior of a 3-sphere is a 4-ball. It is called a 3-sphere because topologically, the surface itself is 3-dimensional, even though it is curved into the 4th dimension. For example, when traveling on a 3-sphere, you can go north and south, east and west, or along a 3rd set of cardinal directions. This means that a 3-sphere is an example of a 3-manifold. In coordinates, a 3-sphere with center (C0, C1, C2, C3) and radius r is the set of all points (x0, x1, x2, x3) in real, 4-dimensional space (R4) such that ∑ i = 0 3 (x i − C i) 2 = (x 0 − C 0) 2 + (x 1 − C 1) 2 + (x 2 − C 2) 2 + (x 3 − C 3) 2 = r 2 . {\displaystyle \sum _{i=0}^{3}(x_{i}-C_{i})^{2}=(x_{0}-C_{0})^{2}+(x_{1}-C_{1})^{2}+(x_{2}-C_{2})^{2}+(x_{3}-C_{3})^{2}=r^{2}.} The 3-sphere centered at the origin with radius 1 is called the unit 3-sphere and is usually denoted S3: S 3 = { (x 0 , x 1 , x 2 , x 3) ∈ R 4 : x 0 2 + x 1 2 + x 2 2 + x 3 2 = 1 } . {\displaystyle S^{3}=\left\{(x_{0},x_{1},x_{2},x_{3})\in \mathbb {R} ^{4}:x_{0}^{2}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=1\right\}.} It is often convenient to regard R4 as the space with 2 complex dimensions (C2) or the quaternions (H). The unit 3-sphere is then given by S 3 = { (z 1 , z 2) ∈ C 2 : | z 1 | 2 + | z 2 | 2 = 1 } {\displaystyle S^{3}=\left\{(z_{1},z_{2})\in \mathbb {C} ^{2}:|z_{1}|^{2}+|z_{2}|^{2}=1\right\}} or S 3 = { q ∈ H : ‖ q ‖ = 1 } . {\displaystyle S^{3}=\left\{q\in \mathbb {H} :\|q\|=1\right\}.} This description as the quaternions of norm one identifies the 3-sphere with the versors in the quaternion division ring. Just as the unit circle is important for planar polar coordinates, so the 3-sphere is important in the polar view of 4-space involved in quaternion multiplication. See polar decomposition of a quaternion for details of this development of the three-sphere. This view of the 3-sphere is the basis for the study of elliptic space as developed by Georges Lemaître.[1] The 3-dimensional surface volume of a 3-sphere of radius r is S V = 2 π 2 r 3 {\displaystyle SV=2\pi ^{2}r^{3}\,} while the 4-dimensional hypervolume (the content of the 4-dimensional region, or ball, bounded by the 3-sphere) is H = 1 2 π 2 r 4 . {\displaystyle H={\frac {1}{2}}\pi ^{2}r^{4}.} Every non-empty intersection of a 3-sphere with a three-dimensional hyperplane is a 2-sphere (unless the hyperplane is tangent to the 3-sphere, in which case the intersection is a single point). As a 3-sphere moves through a given three-dimensional hyperplane, the intersection starts out as a point, then becomes a growing 2-sphere that reaches its maximal size when the hyperplane cuts right through the "equator" of the 3-sphere. Then the 2-sphere shrinks again down to a single point as the 3-sphere leaves the hyperplane. In a given three-dimensional hyperplane, a 3-sphere can rotate about an "equatorial plane" (analogous to a 2-sphere rotating about a central axis), in which case it appears to be a 2-sphere whose size is constant. A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply connected. What this means, in the broad sense, is that any loop, or circular path, on the 3-sphere can be continuously shrunk to a point without leaving the 3-sphere. The Poincaré conjecture, proved in 2003 by Grigori Perelman, provides that the 3-sphere is the only three-dimensional manifold (up to homeomorphism) with these properties. The 3-sphere is homeomorphic to the one-point compactification of R3. In general, any topological space that is homeomorphic to the 3-sphere is called a topological 3-sphere. The homology groups of the 3-sphere are as follows: H0(S3, Z) and H3(S3, Z) are both infinite cyclic, while Hi(S3, Z) = {} for all other indices i. Any topological space with these homology groups is known as a homology 3-sphere. Initially Poincaré conjectured that all homology 3-spheres are homeomorphic to S3, but then he himself constructed a non-homeomorphic one, now known as the Poincaré homology sphere. Infinitely many homology spheres are now known to exist. For example, a Dehn filling with slope 1/n on any knot in the 3-sphere gives a homology sphere; typically these are not homeomorphic to the 3-sphere. As to the homotopy groups, we have π1(S3) = π2(S3) = {} and π3(S3) is infinite cyclic. The higher-homotopy groups (k ≥ 4) are all finite abelian but otherwise follow no discernible pattern. For more discussion see homotopy groups of spheres. Homotopy groups of S3 k 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 πk(S3) 0 0 0 Z Z2 Z2 Z12 Z2 Z2 Z3 Z15 Z2 Z2⊕Z2 Z12⊕Z2 Z84⊕Z2⊕Z2 Z2⊕Z2 Z6 The 3-sphere is naturally a smooth manifold, in fact, a closed embedded submanifold of R4. The Euclidean metric on R4 induces a metric on the 3-sphere giving it the structure of a Riemannian manifold. As with all spheres, the 3-sphere has constant positive sectional curvature equal to 1/r2 where r is the radius. Much of the interesting geometry of the 3-sphere stems from the fact that the 3-sphere has a natural Lie group structure given by quaternion multiplication (see the section below on group structure). The only other spheres with such a structure are the 0-sphere and the 1-sphere (see circle group). Unlike the 2-sphere, the 3-sphere admits nonvanishing vector fields (sections of its tangent bundle). One can even find three linearly independent and nonvanishing vector fields. These may be taken to be any left-invariant vector fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial. For a general discussion of the number of linear independent vector fields on a n-sphere, see the article vector fields on spheres. There is an interesting action of the circle group T on S3 giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of S3 as a subset of C2, the action is given by (z 1 , z 2) ⋅ λ = (z 1 λ , z 2 λ) ∀ λ ∈ T {\displaystyle (z_{1},z_{2})\cdot \lambda =(z_{1}\lambda ,z_{2}\lambda)\quad \forall \lambda \in \mathbb {T} } . The orbit space of this action is homeomorphic to the two-sphere S2. Since S3 is not homeomorphic to S2 × S1, the Hopf bundle is nontrivial. There are several well-known constructions of the three-sphere. Here we describe gluing a pair of three-balls and then the one-point compactification. A 3-sphere can be constructed topologically by "gluing" together the boundaries of a pair of 3-balls. The boundary of a 3-ball is a 2-sphere, and these two 2-spheres are to be identified. That is, imagine a pair of 3-balls of the same size, then superpose them so that their 2-spherical boundaries match, and let matching pairs of points on the pair of 2-spheres be identically equivalent to each other. In analogy with the case of the 2-sphere (see below), the gluing surface is called an equatorial sphere. Note that the interiors of the 3-balls are not glued to each other. One way to think of the fourth dimension is as a continuous real-valued function of the 3-dimensional coordinates of the 3-ball, perhaps considered to be "temperature". We take the "temperature" to be zero along the gluing 2-sphere and let one of the 3-balls be "hot" and let the other 3-ball be "cold". The "hot" 3-ball could be thought of as the "upper hemisphere" and the "cold" 3-ball could be thought of as the "lower hemisphere". The temperature is highest/lowest at the centers of the two 3-balls. This construction is analogous to a construction of a 2-sphere, performed by gluing the boundaries of a pair of disks. A disk is a 2-ball, and the boundary of a disk is a circle (a 1-sphere). Let a pair of disks be of the same diameter. Superpose them and glue corresponding points on their boundaries. Again one may think of the third dimension as temperature. Likewise, we may inflate the 2-sphere, moving the pair of disks to become the northern and southern hemispheres. After removing a single point from the 2-sphere, what remains is homeomorphic to the Euclidean plane. In the same way, removing a single point from the 3-sphere yields three-dimensional space. An extremely useful way to see this is via stereographic projection. We first describe the lower-dimensional version. Rest the south pole of a unit 2-sphere on the xy-plane in three-space. We map a point P of the sphere (minus the north pole N) to the plane by sending P to the intersection of the line NP with the plane. Stereographic projection of a 3-sphere (again removing the north pole) maps to three-space in the same manner. (Notice that, since stereographic projection is conformal, round spheres are sent to round spheres or to planes.) A somewhat different way to think of the one-point compactification is via the exponential map. Returning to our picture of the unit two-sphere sitting on the Euclidean plane: Consider a geodesic in the plane, based at the origin, and map this to a geodesic in the two-sphere of the same length, based at the south pole. Under this map all points of the circle of radius π are sent to the north pole. Since the open unit disk is homeomorphic to the Euclidean plane, this is again a one-point compactification. The exponential map for 3-sphere is similarly constructed; it may also be discussed using the fact that the 3-sphere is the Lie group of unit quaternions. The four Euclidean coordinates for S3 are redundant since they are subject to the condition that x02 + x12 + x22 + x32 = 1. As a 3-dimensional manifold one should be able to parameterize S3 by three coordinates, just as one can parameterize the 2-sphere using two coordinates (such as latitude and longitude). Due to the nontrivial topology of S3 it is impossible to find a single set of coordinates that cover the entire space. Just as on the 2-sphere, one must use at least two coordinate charts. Some different choices of coordinates are given below. It is convenient to have some sort of hyperspherical coordinates on S3 in analogy to the usual spherical coordinates on S2. One such choice — by no means unique — is to use (ψ, θ, φ), where x 0 = r cos ⁡ ψ x 1 = r sin ⁡ ψ cos ⁡ θ x 2 = r sin ⁡ ψ sin ⁡ θ cos ⁡ φ x 3 = r sin ⁡ ψ sin ⁡ θ sin ⁡ φ {\displaystyle {\begin{aligned}x_{0}&=r\cos \psi \\x_{1}&=r\sin \psi \cos \theta \\x_{2}&=r\sin \psi \sin \theta \cos \varphi \\x_{3}&=r\sin \psi \sin \theta \sin \varphi \end{aligned}}} where ψ and θ run over the range 0 to π, and φ runs over 0 to 2π. Note that, for any fixed value of ψ, θ and φ parameterize a 2-sphere of radius r sin ⁡ ψ {\displaystyle r\sin \psi } , except for the degenerate cases, when ψ equals 0 or π, in which case they describe a point. The round metric on the 3-sphere in these coordinates is given by[2] d s 2 = r 2 [d ψ 2 + sin 2 ⁡ ψ (d θ 2 + sin 2 ⁡ θ d φ 2)] {\displaystyle ds^{2}=r^{2}\left[d\psi ^{2}+\sin ^{2}\psi \left(d\theta ^{2}+\sin ^{2}\theta \,d\varphi ^{2}\right)\right]} and the volume form by d V = r 3 (sin 2 ⁡ ψ sin ⁡ θ) d ψ ∧ d θ ∧ d φ . {\displaystyle dV=r^{3}\left(\sin ^{2}\psi \,\sin \theta \right)\,d\psi \wedge d\theta \wedge d\varphi .} These coordinates have an elegant description in terms of quaternions. Any unit quaternion q can be written as a versor: q = e τ ψ = cos ⁡ ψ + τ sin ⁡ ψ {\displaystyle q=e^{\tau \psi }=\cos \psi +\tau \sin \psi } where τ is a unit imaginary quaternion; that is, a quaternion that satisfies τ2 = −1. This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in Im H so any such τ can be written: τ = (cos ⁡ θ) i + (sin ⁡ θ cos ⁡ φ) j + (sin ⁡ θ sin ⁡ φ) k {\displaystyle \tau =(\cos \theta)i+(\sin \theta \cos \varphi)j+(\sin \theta \sin \varphi)k} With τ in this form, the unit quaternion q is given by q = e τ ψ = x 0 + x 1 i + x 2 j + x 3 k {\displaystyle q=e^{\tau \psi }=x_{0}+x_{1}i+x_{2}j+x_{3}k} where x0,1,2,3 are as above. When q is used to describe spatial rotations (cf. quaternions and spatial rotations), it describes a rotation about τ through an angle of 2ψ. The Hopf fibration can be visualized using a stereographic projection of S3 to R3 and then compressing R3 to a ball. This image shows points on S2 and their corresponding fibers with the same color. For unit radius another choice of hyperspherical coordinates, (η, ξ1, ξ2), makes use of the embedding of S3 in C2. In complex coordinates (z1, z2) ∈ C2 we write z 1 = e i ξ 1 sin ⁡ η z 2 = e i ξ 2 cos ⁡ η . {\displaystyle {\begin{aligned}z_{1}&=e^{i\,\xi _{1}}\sin \eta \\z_{2}&=e^{i\,\xi _{2}}\cos \eta .\end{aligned}}} This could also be expressed in R4 as x 0 = cos ⁡ ξ 1 sin ⁡ η x 1 = sin ⁡ ξ 1 sin ⁡ η x 2 = cos ⁡ ξ 2 cos ⁡ η x 3 = sin ⁡ ξ 2 cos ⁡ η . {\displaystyle {\begin{aligned}x_{0}&=\cos \xi _{1}\sin \eta \\x_{1}&=\sin \xi _{1}\sin \eta \\x_{2}&=\cos \xi _{2}\cos \eta \\x_{3}&=\sin \xi _{2}\cos \eta .\end{aligned}}} Here η runs over the range 0 to π/2, and ξ1 and ξ2 can take any values between 0 and 2π. These coordinates are useful in the description of the 3-sphere as the Hopf bundle S 1 → S 3 → S 2 . {\displaystyle S^{1}\to S^{3}\to S^{2}.\,} A diagram depicting the poloidal (ξ1) direction, represented by the red arrow, and the toroidal (ξ2) direction, represented by the blue arrow, although the terms poloidal and toroidal are arbitrary in this flat torus case. For any fixed value of η between 0 and π/2, the coordinates (ξ1, ξ2) parameterize a 2-dimensional torus. Rings of constant ξ1 and ξ2 above form simple orthogonal grids on the tori. See image to right. In the degenerate cases, when η equals 0 or π/2, these coordinates describe a circle. The round metric on the 3-sphere in these coordinates is given by d s 2 = d η 2 + sin 2 ⁡ η d ξ 1 2 + cos 2 ⁡ η d ξ 2 2 {\displaystyle ds^{2}=d\eta ^{2}+\sin ^{2}\eta \,d\xi _{1}^{2}+\cos ^{2}\eta \,d\xi _{2}^{2}} and the volume form by d V = sin ⁡ η cos ⁡ η d η ∧ d ξ 1 ∧ d ξ 2 . {\displaystyle dV=\sin \eta \cos \eta \,d\eta \wedge d\xi _{1}\wedge d\xi _{2}.} To get the interlocking circles of the Hopf fibration, make a simple substitution in the equations above[3] z 1 = e i (ξ 1 + ξ 2) sin ⁡ η z 2 = e i (ξ 2 − ξ 1) cos ⁡ η . {\displaystyle {\begin{aligned}z_{1}&=e^{i\,(\xi _{1}+\xi _{2})}\sin \eta \\z_{2}&=e^{i\,(\xi _{2}-\xi _{1})}\cos \eta .\end{aligned}}} In this case η, and ξ1 specify which circle, and ξ2 specifies the position along each circle. One round trip (0 to 2π) of ξ1 or ξ2 equates to a round trip of the torus in the 2 respective directions. Another convenient set of coordinates can be obtained via stereographic projection of S3 from a pole onto the corresponding equatorial R3 hyperplane. For example, if we project from the point (−1, 0, 0, 0) we can write a point p in S3 as p = (1 − ‖ u ‖ 2 1 + ‖ u ‖ 2 , 2 u 1 + ‖ u ‖ 2) = 1 + u 1 − u {\displaystyle p=\left({\frac {1-\|u\|^{2}}{1+\|u\|^{2}}},{\frac {2\mathbf {u} }{1+\|u\|^{2}}}\right)={\frac {1+\mathbf {u} }{1-\mathbf {u} }}} where u = (u1, u2, u3) is a vector in R3 and ‖u‖2 = u12 + u22 + u32. In the second equality above, we have identified p with a unit quaternion and u = u1i + u2j + u3k with a pure quaternion. (Note that the numerator and denominator commute here even though quaternionic multiplication is generally noncommutative). The inverse of this map takes p = (x0, x1, x2, x3) in S3 to u = 1 1 + x 0 (x 1 , x 2 , x 3) . {\displaystyle \mathbf {u} ={\frac {1}{1+x_{0}}}\left(x_{1},x_{2},x_{3}\right).} We could just as well have projected from the point (1, 0, 0, 0), in which case the point p is given by p = (− 1 + ‖ v ‖ 2 1 + ‖ v ‖ 2 , 2 v 1 + ‖ v ‖ 2) = − 1 + v 1 + v {\displaystyle p=\left({\frac {-1+\|v\|^{2}}{1+\|v\|^{2}}},{\frac {2\mathbf {v} }{1+\|v\|^{2}}}\right)={\frac {-1+\mathbf {v} }{1+\mathbf {v} }}} where v = (v1, v2, v3) is another vector in R3. The inverse of this map takes p to v = 1 1 − x 0 (x 1 , x 2 , x 3) . {\displaystyle \mathbf {v} ={\frac {1}{1-x_{0}}}\left(x_{1},x_{2},x_{3}\right).} Note that the u coordinates are defined everywhere but (−1, 0, 0, 0) and the v coordinates everywhere but (1, 0, 0, 0). This defines an atlas on S3 consisting of two coordinate charts or "patches", which together cover all of S3. Note that the transition function between these two charts on their overlap is given by v = 1 ‖ u ‖ 2 u {\displaystyle \mathbf {v} ={\frac {1}{\|u\|^{2}}}\mathbf {u} } and vice versa. When considered as the set of unit quaternions, S3 inherits an important structure, namely that of quaternionic multiplication. Because the set of unit quaternions is closed under multiplication, S3 takes on the structure of a group. Moreover, since quaternionic multiplication is smooth, S3 can be regarded as a real Lie group. It is a nonabelian, compact Lie group of dimension 3. When thought of as a Lie group, S3 is often denoted Sp(1) or U(1, H). It turns out that the only spheres that admit a Lie group structure are S1, thought of as the set of unit complex numbers, and S3, the set of unit quaternions (The degenerate case S0 which consists of the real numbers 1 and −1 is also a Lie group, albeit a 0-dimensional one). One might think that S7, the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is nonassociative. The octonionic structure does give S7 one important property: parallelizability. It turns out that the only spheres that are parallelizable are S1, S3, and S7. By using a matrix representation of the quaternions, H, one obtains a matrix representation of S3. One convenient choice is given by the Pauli matrices: x 1 + x 2 i + x 3 j + x 4 k ↦ (x 1 + i x 2 x 3 + i x 4 − x 3 + i x 4 x 1 − i x 2) . {\displaystyle x_{1}+x_{2}i+x_{3}j+x_{4}k\mapsto {\begin{pmatrix}\;\;\,x_{1}+ix_{2}&x_{3}+ix_{4}\\-x_{3}+ix_{4}&x_{1}-ix_{2}\end{pmatrix}}.} This map gives an injective algebra homomorphism from H to the set of 2 × 2 complex matrices. It has the property that the absolute value of a quaternion q is equal to the square root of the determinant of the matrix image of q. The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group SU(2). Thus, S3 as a Lie group is isomorphic to SU(2). Using our Hopf coordinates (η, ξ1, ξ2) we can then write any element of SU(2) in the form (e i ξ 1 sin ⁡ η e i ξ 2 cos ⁡ η − e − i ξ 2 cos ⁡ η e − i ξ 1 sin ⁡ η) . {\displaystyle {\begin{pmatrix}e^{i\,\xi _{1}}\sin \eta &e^{i\,\xi _{2}}\cos \eta \\-e^{-i\,\xi _{2}}\cos \eta &e^{-i\,\xi _{1}}\sin \eta \end{pmatrix}}.} Another way to state this result is if we express the matrix representation of an element of SU(2) as an exponential of a linear combination of the Pauli matrices. It is seen that an arbitrary element U ∈ SU(2) can be written as U = exp ⁡ (∑ i = 1 3 α i J i) . {\displaystyle U=\exp \left(\sum _{i=1}^{3}\alpha _{i}J_{i}\right).} [4] The condition that the determinant of U is +1 implies that the coefficients α1 are constrained to lie on a 3-sphere. In Edwin Abbott Abbott's Flatland, published in 1884, and in Sphereland, a 1965 sequel to Flatland by Dionys Burger, the 3-sphere is referred to as an oversphere, and a 4-sphere is referred to as a hypersphere. Writing in the American Journal of Physics,[5] Mark A. Peterson describes three different ways of visualizing 3-spheres and points out language in The Divine Comedy that suggests Dante viewed the Universe in the same way; Carlo Rovelli supports the same idea.[6] In Art Meets Mathematics in the Fourth Dimension,[7] Stephen L. Lipscomb develops the concept of the hypersphere dimensions as it relates to art, architecture, and mathematics. 1-sphere, 2-sphere, n-sphere tesseract, polychoron, simplex Pauli matrices Hopf bundle, Riemann sphere Poincaré sphere Reeb foliation Clifford torus ^ Lemaître, Georges (1948). "Quaternions et espace elliptique". Acta. 12. Pontifical Academy of Sciences: 57–78. ^ Landau, Lev D.; Lifshitz, Evgeny M. (1988). Classical Theory of Fields. 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Weeks, Jeffrey R. (1985). "Chapter 14: The Hypersphere". The Shape of Space: How to Visualize Surfaces and Three-dimensional Manifolds. A Warning on terminology: Our two-sphere is defined in three-dimensional space, where it is the boundary of a three-dimensional ball. This terminology is standard among mathematicians, but not among physicists. So don't be surprised if you find people calling the two-sphere a three-sphere. Zamboj, Michal (8 Jan 2021). "Synthetic construction of the Hopf fibration in a double orthogonal projection of 4-space". Journal of Computational Design and Engineering. 8 (3): 836–854. arXiv:2003.09236v2. doi:10.1093/jcde/qwab018. Weisstein, Eric W. "Hypersphere". MathWorld. Note: This article uses the alternate naming scheme for spheres in which a sphere in n-dimensional space is termed an n-sphere. Retrieved from " 3 The following pages link to 3-sphere External tools (link count transclusion count sorted list) · See help page for transcluding these entries Showing 50 items. View (previous 50 | next 50) (20 | 50 | 100 | 250 | 500)Cosmic inflation (links | edit) Conjecture (links | edit) Differential topology (links | edit) Diffeomorphism (links | edit) Euclidean geometry (links | edit) Euler's formula (links | edit) Poincaré conjecture (links | edit) 4-polytope (links | edit) Qubit (links | edit) Sphere (links | edit) Tetrahedron (links | edit) Tesseract (links | edit) Universe (links | edit) Simplex (links | edit) N-sphere (links | edit) Quaternion (links | edit) Tangloids (links | edit) Ensemble (mathematical physics) (links | edit) Unknot (links | edit) Torus (links | edit) Shape of the universe (links | edit) Ball (mathematics) (links | edit) Unit vector (links | edit) Orthogonal group (links | edit) 3D rotation group (links | edit) Symplectic group (links | edit) Special unitary group (links | edit) List of unsolved problems in mathematics (links | edit) Quaternions and spatial rotation (links | edit) Spherical harmonics (links | edit) Orders of magnitude (length) (links | edit) Homotopy (links | edit) Geometrization conjecture (links | edit) Hsiang–Lawson's conjecture (links | edit) Grigori Perelman (links | edit) Thurston elliptization conjecture (links | edit) Elliptic geometry (links | edit) List of mathematical shapes (links | edit) 24-cell (links | edit) Peter–Weyl theorem (links | edit) Instanton (links | edit) List of geometric topology topics (links | edit) CW complex (links | edit) Knot (mathematics) (links | edit) Foliation (links | edit) Trefoil knot (links | edit) Charts on SO(3) (links | edit) T-duality (links | edit) Linking number (links | edit) 600-cell (links | edit) View (previous 50 | next 50) (20 | 50 | 100 | 250 | 500) Retrieved from " WhatLinksHere/3-sphere" In math, a conjecture is like a smart guess — something we think is true but haven't proven. If someone finds an example that shows the guess is wrong, that's a counterexample. It's a bit like playing a detective game in mathematics. In this guide, we'll look at these two ideas, breaking them down in easy-to-understand terms. Understanding Conjectures:A conjecture is an unproven statement that is believed to be true based on observations. Conjectures arise from patterns noticed by mathematicians. While some conjectures have been proven, others remain unproven and open to exploration. Recognizing Counterexamples:A counterexample is a specific case or instance that disproves a conjecture or statement. If even one counterexample exists, it means the conjecture is not universally true. Counterexamples are indispensable in mathematics for several reasons: They refine and correct conjectures. They prevent mathematicians from pursuing false statements. They offer clarity on the limitations of a statement’s accuracy. Example 1: Conjecture about Prime NumbersConjecture: “All numbers less than \(10\) are prime.” Solution:Considering the numbers less than \(10\): \(2\), \(3\), \(4\), \(5\), \(6\), \(7\), \(8\), and \(9\), we can identify counterexamples. Numbers \(4\), \(6\), \(8\), and \(9\) are not prime. Hence, the conjecture is false. Example 2: Conjecture about Even NumbersConjecture: “All even numbers greater than \(2\) are not prime.” Solution:For this conjecture, the number \(2\) is a counterexample. \(2\) is even and is also prime. While the statement does mention numbers greater than \(2\), the inclusion of \(2\) as an even prime number highlights the need for precision in the formulation of conjectures. Example 3: Fermat’s Last TheoremConjecture: There are no three integers \(a\), \(b\), and \(c\) that can satisfy the equation \(a^n + b^n = c^n\) for any integer value of \(n\) greater than \(2\). Solution:This conjecture remained unproven for centuries. However, it was eventually proven true by Andrew Wiles in 1994, meaning there are no counterexamples. Conjecture: All positive integers are greater than \(0\). Is this true or false? If false, provide a counterexample. Conjecture: The square of any integer is positive. Is this true or false? If false, provide a counterexample. Conjecture: All birds can fly. Is this true or false? If false, provide a counterexample. Answers: True. By definition, positive integers are greater than \(0\). True. The square of any integer, whether positive or negative, is always positive. False. Counterexample: Ostriches, penguins, and kiwis are birds that cannot fly.